

Port-based teleportation in arbitrary dimension

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December 30, 2016

Abstract

We study port-based teleportation protocols and fully characterize their performance for arbitrary dimensions and number of ports. We develop new mathematical tools to study the symmetries of the measurement operators that arise in these protocols and belong to the algebra of partially transposed operators. First, we introduce the theory of the partially reduced irreducible representations which provides an elegant way of understanding the properties of subsystems of a large system with general symmetries. We then use this theory to obtain a simple representation of the algebra of partially transposed operators and thus explicitly determine the fidelity and the probability of success of any port-based teleportation scheme.

1 Introduction

Quantum teleportation is one of the most important primitives in the Quantum Information Processing [2]. This technique allows to transfer the state of an unknown quantum system from the sender to the receiver without the need to exchange a physical system. It has led to a large number of theoretical advances in quantum information theory and quantum computing. [3, 6, 7, 12, 13, 15, 16].

The first teleportation protocol involved two parties, Alice and Bob, each sharing a half of the maximally entangled state [2]. We will further refer to it as a ‘resource state’. Alice wants to send an (unknown) state of a subsystem in her possession to Bob. She performs a projective measurement on her subsystem and the half of the maximally entangled state and communicates its classical outcome to Bob. He then reliably recovers the state which Alice communicated by applying a unitary correction operation conditioned on Alice’s message.

A different teleportation protocol was introduced by Knill, Laflamme and Milburn [13]. Alice and Bob shared a large multipartite entangled state which is markedly different from a single maximally entangled resource state used in the original teleportation protocol. It followed the same sequence of steps: projective measurement, communication of the classical outcome to Bob, and unitary correction, and possessed two interesting properties. First, the measurements consisting of the Fourier transform followed by the projective measurement are implementable by linear optical elements. Secondly, the protocol yields a non-unit success probability even with the optimal resource state, and tends to one as the dimension of the resource state increases [8].

The above protocols and their variants followed the same 3-step procedure: 1) measurement, 2) communication of the classical outcome, and 3) unitary correction that precluded it from being composable. In 2008, a breakthrough result from Ishizaka and Hiroshima introduced a novel port-based teleportation protocol (PBT) which does not require the last step in the sequence [10]. Parties share a large resource state consisting of N copies of the maximally entangled states $|\Psi^-\rangle^{\otimes N}$, where each singlet is a two-qubit state. Each copy of $|\Psi^-\rangle$ is called a port. Alice performs a joint measurement \mathcal{X} on her half of the resource state together with the unknown state θ which she wishes to teleport and communicates the result to Bob. The outcome of the measurement signifies the port in which the state has been teleported to. In order to obtain the teleported state Bob discards all ports except for the one indicated by Alice's outcome. There are two versions of the PBT protocol, depending on the exact set of measurements used by Alice. The first type, *deterministic* teleportation is described by the set of N POVM elements $\mathcal{X} = \{\Pi_a\}_{a=1}^N$. Upon measuring a -th element the teleported state ends up in the a -th port on Bob's side. He thus traces out all but a -th subsystem which contains the teleported state. The second type, *probabilistic* PBT, consists of a measurement with $N + 1$ POVM elements $\{\Pi_a\}_{a=0}^N$, where X_0 indicates a failure of the teleportation. In this protocol, when Alice obtains the input $a \in \{1, \dots, N\}$, the parties proceed as above. When she obtains 0, then they abort the protocol.

In the probabilistic PBT the state always gets teleported to Bob, but it suffers noise during the process. The fidelity of the teleported state tends to 1 as the number of ports $N \rightarrow \infty$. In the second protocol, when Alice obtains $a \in \{1, \dots, N\}$ the state always gets teleported to Bob with perfect fidelity, but with some probability (which vanishes in the limit $N \rightarrow \infty$) Alice aborts.

PBT schemes found novel applications in the areas where the existing teleportation schemes fell short of. They provide new architecture for the universal programmable quantum processor performing computation by teleportation with the property of it being composable. [11].

In position-based cryptography, PBT schemes were used to engineer new attacks on the cryptographic primitives exponentially reducing the amount of consumable entanglement in the worst case. This enabled to render a large number of position-based cryptographic schemes insecure if the attacker possessed an exponential amount of entanglement (down from doubly exponential) in the parameters of the scheme [1]. This extends to the implementation of the instantaneous computation. The existing schemes for such computations that required correction after the teleportation resulted in the exponential blow-up of the entanglement consumption due to the fact that one had to account for the every single correction in order to obtain the result of the computation.

Recently, the composable nature of the qubit PBT schemes made it possible to connect the field of communication complexity and a Bell inequality violation [4]. It allowed to show that any quantum advantage obtained by a protocol for an arbitrary communication complexity problem resulted in the violation of a Bell inequality, certifying the quantum nature of the advantage.

A full solution to the qubit PBT schemes can be used to obtain the performance of the square-root measurement schemes for mixed states obtaining explicit probabilities of success when the set of states to be discriminated has certain symmetries and POVMs are of nearly maximal rank [1].

Evaluating the performance of the PBT is tantamount to determining the spectral properties of the measurement operators \mathcal{X} . To study them authors in [10] treated $N + 1$ qubits as spins, recursively building a basis for constituents of \mathcal{X} and using the Glebsch-Gordan (CG) coefficients determined their eigenvalues with the painstaking amount of effort. This approach has been successful for studying systems of $N + 1$ qubits and relied on the existence of the closed form for the CG coefficients and therefore was limited to $SU(2)^{\otimes N}$. In the case of $SU(d)^{\otimes N}$, with $d > 2$

there exists no closed form of the CG coefficients and thus it is impossible to obtain the spectrum of ρ without incurring an exponential overhead in d when building the basis recursively.

Recent work by [19] provides a way to obtain explicit closed-form expressions for the fidelity and success probability of PBT using graphical algebra which is a variant of Temperley-Lieb algebra for an arbitrary d and $N \in \{2, 3, 4\}$. Here too the obtained expressions contain the number of different terms which grow exponentially in N making it computationally intractable to evaluate the performance of even a modestly-sized PBT scheme.

In our work we develop new mathematical tools to study the symmetries of \mathcal{X} which enable us to efficiently evaluate the performance of *any* PBT scheme. Our first contribution is the theory of the partially reduced irreducible representations (PRIR). They provide an elegant way of understanding the properties of subsystems of a large system which has general symmetries. We further use these techniques to provide a simple way to approach to the representation of the algebra of the partially transposed operators. It turns out that the operators describing measurements in any PBT scheme possesses the exact symmetries that make it the element of this algebra. Exploring these symmetries in a principled way makes it possible to understand the structure and the properties of the teleportation.

We characterize the performance of main PBT schemes providing exact expressions for the fidelity of the teleportation and the probability of success in the deterministic and probabilistic schemes, respectively whenever resource state is a maximally entangled state. Moreover, we provide a detailed description of the spectral properties of the POVMs and show how systems that possess similar symmetries can be analyzed using our tools.

2 Structure of the port based teleportation operator

Consider a probabilistic protocol defined by a d -dimensional maximally entangled resource state $|\Phi_d^+\rangle_{AB}$ where $A(B) = A_1(B_1) \dots A_N(B_N)$ set of POVMs $\mathcal{X} = \{\Pi_a\}_{a=1}^N$, where each $\Pi_a = \rho^{-\frac{1}{2}} \varrho_a \rho^{-\frac{1}{2}}$, and

$$\rho = \sum_{a=1}^N \varrho_a, \quad (1)$$

$$\varrho_a = \frac{1}{d^N} P_{CA_a}^+ \otimes \mathbf{1}_{\overline{A_a}}, \quad (2)$$

for $a = 1, \dots, N$. Alice wishes to teleport a qudit θ_C to Bob. The operator $P_{CA_a}^+$ denotes a unnormalised projection onto $|\Phi^+\rangle$ state between systems C and A_a , and $\mathbf{1}_{\overline{A_a}}$ is identity operator on all subsystems A except A_a . The schematic representation is shown in Fig 1.

The key to quantifying the effect of the measurement \mathcal{X} is to understand the spectral structure of ρ . Therefore, we first recast ρ in a different form. First, observe that every projector $P_{CA_a}^+$ can be written as

$$P_{CA_a}^+ = V_{(CA_a)}^{t_C}, \quad (3)$$

where $V_{(CA_a)}$ denotes a permutation operator between systems C and A_a , and t_C denotes a transposition with respect to subsystem C . Therefore, ρ from Eqn. (1) can be written in terms of partially transposed permutation operators:

$$\rho = \frac{1}{d^N} \sum_{a=1}^N V_{(CA_a)}^{t_C} \otimes \mathbf{1}_{\overline{A_a}}. \quad (4)$$

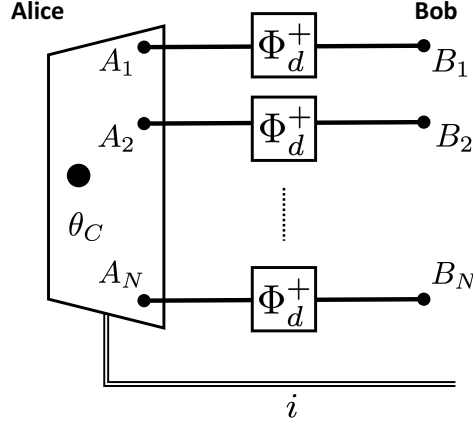


Figure 1: Schematic description of PBT in the arbitrary dimension.

Every element $V_{(CA_a)}^{t_C} \otimes \mathbf{1}_{\overline{A_a}}$ acts as a permutation operator on full $n = N + 1$ (we will use n and N interchangeably) particle Hilbert space $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$. Whenever it is clear from the context we denote every operator $V_{(CA_a)}^{t_C} \otimes \mathbf{1}_{\overline{A_a}}$ just by $V_{(CA_a)}^{t_C}$. The above form enables us to identify ρ as the element of a recently studied algebra of partially transposed permutation operators $\mathcal{A}_n^{t_n}(d)$ acting permutationally in the space $(\mathbb{C}^d)^{\otimes n}$, where $d \in \mathbb{N}$ and $d \geq 2$ [14, 18]. It turns out that $\mathcal{A}_n^{t_n}(d)$ decomposes into a direct sum of two types of ideals:

$$\mathcal{A}_n^{t_n}(d) = \mathcal{M} \oplus \mathcal{S}. \quad (5)$$

For our purposes we are only interested in ideal \mathcal{M} which includes the irreducible representations of $\mathcal{A}_n^{t_n}(d)$ indexed by irreducible representations of the group $S(n-2)$ which are strictly connected with the representations of the group $S(n-1)$ [14, 18] (see Fig. 2). To make our further considerations more transparent and have explicit notational connection with the previous analysis of the algebra $\mathcal{A}_n^{t_n}(d)$ we consider operator ρ without factor $1/d^{n-1}$ and we change numeration of the subsystems in such a way, that we can rewrite general form of equation (4) as:

$$\eta = \sum_{a=1}^{n-1} V^{t_n}(a, n), \quad (6)$$

where t_n denotes a partial transposition on the last n^{th} subsystem, and (a, n) is a permutation between subsystems a and n . Here, the role of subsystem C is played by the subsystem n . In order to evaluate the performance of the PBT scheme explicitly one has to obtain full spectral analysis of the operator given in (6).

Theorem 1. Let $\eta = \sum_{a=1}^{n-1} V^{t_n}(a, n-1)$, then η can be written as:

$$\eta = \bigoplus_{\alpha \vdash n-2} \bigoplus_{\substack{\mu \vdash n-1 \\ \mu \in \alpha}} \eta_{\mu}(\alpha) = \bigoplus_{\alpha \vdash n-2} \bigoplus_{\substack{\mu \vdash n-1 \\ \mu \in \alpha}} P_{\mu} \sum_{a=1}^{n-1} V(a, n-1) P_{\alpha} V^{t_n}(n-1, n) V(a, n-1), \quad (7)$$

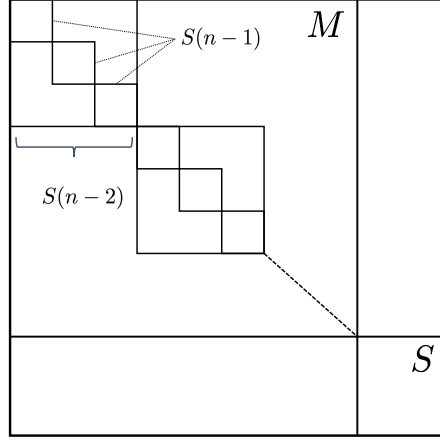


Figure 2: The structure of $\mathcal{A}_n^{tn}(d)$.

where α, μ run over those Young diagrams allowed by dimension of the single system, by $\mu \in \alpha$ we understand such Young diagrams μ which are obtained from α by adding one box in a proper way. The operators $\eta_\mu(\alpha)$ are proportional to projectors, i.e. $\eta_\mu(\alpha) = \gamma_\mu(\alpha)F_\mu(\alpha)$, and $F_\mu(\alpha)$ are projectors of the dimension $\dim F_\mu(\alpha) = d_\mu \tilde{m}_\alpha$, where \tilde{m}_α is the multiplicity of the irrep labelled by α of algebra $\mathcal{A}_n^{tn}(d)$, and d_μ is the dimension of the irrep of $S(n-1)$ labelled by μ . The projectors $F_\mu(\alpha) = M_\alpha P_\mu$, where M_α is projector including multiplicities onto α -th irrep of the algebra $\mathcal{A}_n^{tn}(d)$.

Proof. Consider an arbitrary representation of algebra $\mathcal{A}_n^{tn}(d)$. Let M_α be projector (including multiplicities) onto irrep labelled by $\alpha \vdash n-2$, $S(M_\alpha)$ corresponding subspace, and P_μ be a projector onto irrep of $S(n-1)$ in the same representation (also with multiplicities). Our first goal is to know restriction of the operator η to irrep labelled by α . To obtain desired result we need to express it in terms of operators $\{v_{ij}^{ab}(\alpha)\}$, where $1 \leq a, b \leq n-1$ and $1 \leq i, j \leq d_\alpha$ (see [18])

$$v_{ij}^{ab}(\alpha) = V(a, n-1)E_{ij}^\alpha V^{tn}(n-1, n)V(b, n-1), \quad (8)$$

since they span irrep labelled by α . Using above expression we can decompose operator η in the following manner

$$\begin{aligned} \eta &= \sum_{a=1}^{n-1} V(a, n-1)V^{tn}(n-1, n)V(a, n-1) = \sum_{\alpha} \sum_{a=1}^{n-1} V(a, n-1)P_\alpha V^{tn}(n-1, n)V(a, n-1) \\ &= \sum_{\alpha} \sum_{i=1}^{d_\alpha} \sum_{a=1}^{n-1} V(a, n-1)E_{ii}^\alpha V^{tn}(n-1, n)V(a, n-1) = \sum_{\alpha} \sum_{i=1}^{d_\alpha} \sum_{a=1}^{n-1} v_{ii}^{aa}(\alpha) = \sum_{\alpha} \eta(\alpha). \end{aligned} \quad (9)$$

with

$$\eta(\alpha) = \sum_{\alpha} \sum_{i=1}^{d_\alpha} v_{ii}^{aa}(\alpha) = \sum_{a=1}^{n-1} V(a, n-1)P_\alpha V^{tn}(n-1, n)V(a, n-1), \quad (10)$$

thus $\eta(\alpha) \in S(M_\alpha)$. Operators $\eta(\alpha)$ are supported on the space $S(M_\alpha)$ and invariant under action of $S(n-1)$, hence its eigenprojectors are $F_\mu(\alpha) = M_\alpha P_\mu$, so

$$\eta(\alpha) = \bigoplus_{\substack{\mu \vdash n-1 \\ \mu \in \alpha}} \gamma_\mu(\alpha) M_\alpha P_\mu = \bigoplus_{\mu \in \alpha} \eta_\mu(\alpha), \quad \gamma_\mu(\alpha) \in \mathbb{C}, \quad (11)$$

with $\eta_\mu(\alpha) = P_\mu \eta(\alpha) P_\mu$. Then obviously we have

$$F_\mu(\alpha) = \gamma_\mu^{-1}(\alpha) P_\mu \eta(\alpha) P_\mu. \quad (12)$$

All of the properties are derived from the properties of the underlying algebra, and are thus independent of representation. For any representation $\text{Tr}[P_\mu M_\alpha] = d_\mu \tilde{m}_\alpha$, where \tilde{m}_α is multiplicity of the projector M_α . \square

We illustrate above theorem in the Fig. 3. For our further purposes sometimes we switch to

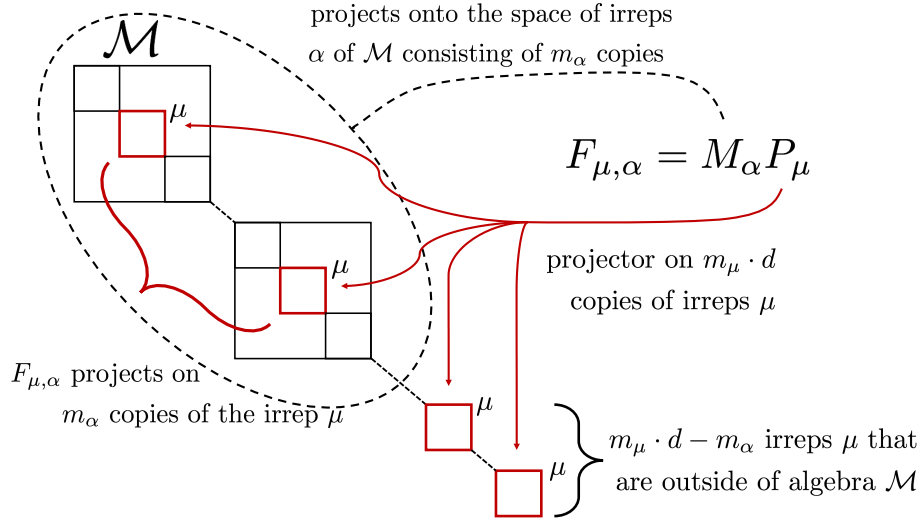


Figure 3: Graphical illustration of the action of the projector $F_{\alpha,\mu}$ in Theorem 1.

natural representation (for example when we want to compute partial trace). Fortunately from [14, 18] we know that $\tilde{m}_\alpha = m_\alpha$, where m_α is multiplicity of irrep labelled by α in natural representation of the group $S(n-2)$. Keeping all notation introduced in the previous theorem we now find the formula for the eigenvalues of the port based teleportation operator.

Proposition 2. *The numbers $\gamma_\mu(\alpha)$ given in equation (12) are eigenvalues of the operator η , and are given by*

$$\gamma_\mu(\alpha) = (n-1) \frac{m_\mu d_\alpha}{m_\alpha d_\mu} \quad (13)$$

or equivalently

$$\gamma_\mu(\alpha) = d + \frac{1}{2}(n-1)(n-2) \frac{\chi^\mu(12)}{d_\mu} - \frac{1}{2}(n-2)(n-3) \frac{\chi^\alpha(12)}{d_\alpha}. \quad (14)$$

By m_α, m_μ we denote multiplicities ¹ in the natural representation, by d_α, d_μ dimensions, and by $\chi^\mu(12), \chi^\alpha(12)$ characters calculated on transpositions (12) of the respective irreps.

Proof. Since by Theorem 1 we have $\eta = \sum_{\mu \vdash n-1} \gamma_\mu(\alpha) F_\mu(\alpha)$, then $\eta_\mu(\alpha) = \gamma_\mu(\alpha) F_\mu(\alpha)$. Thanks to this numbers $\gamma_\mu(\alpha)$ can be expressed as

$$\gamma_\mu(\alpha) = \frac{\text{Tr } \eta_\mu(\alpha)}{\text{Tr } F_\mu(\alpha)} = \frac{\text{Tr } \eta_\mu(\alpha)}{d_\mu m_\alpha}. \quad (15)$$

In the last step we have to compute $\text{Tr } \eta_\mu(\alpha)$. Taking into account decomposition given through equation (7) and Fact 16 from Appendix C we write

$$\text{Tr } \eta_\mu(\alpha) = (n-1) \text{Tr} [P_\mu P_\alpha V^{t_n}(n-1, n)] = \text{Tr} [P_\mu (P_\alpha \otimes \mathbf{1})] = (n-1) m_\mu d_\alpha. \quad (16)$$

Combining (16) with (15) we obtain first statement of the proposition given in equation (13). To prove expression (14) we consider quantity:

$$\gamma_\mu(\alpha) = \frac{n-1}{d_\mu m_\alpha} \text{Tr} [P_\mu (P_\alpha \otimes \mathbf{1})]. \quad (17)$$

Making use of Fact 15 when analysing P_μ , after few transformations we get

$$\begin{aligned} \gamma_\mu(\alpha) &= \frac{n-1}{m_\alpha(n-1)!} \sum_{a=1}^{n-1} \sum_{i,j=1}^{d_\mu} \varphi_{ij}^\mu(a, n-1) \text{Tr} [V(a, n-1) F_{ij}^\mu (P_\alpha \otimes \mathbf{1})] \\ &= \frac{n-1}{m_\alpha(n-1)!} \sum_{a=1}^{n-1} \sum_{i,j=1}^{d_\mu} d^{\delta_{a,n-1}} \varphi_{ij}^\mu(a, n-1) \text{Tr} [F_{ij}^\mu P_\alpha]. \end{aligned} \quad (18)$$

From Eqn. (77) and the proof of Fact 15 and property, that $\text{Tr} [E_{ij}^\beta P_\alpha] = \sum_k \text{Tr} [E_{ij}^\beta E_{kk}^\alpha] = \text{Tr} [E_{ij}^\alpha] = \delta_{ij} m_\alpha$, we write result in the PRIR notation defined in Appendix B

$$\gamma_\mu(\alpha) = \frac{1}{d_\alpha} \sum_{a=1}^{n-1} d^{\delta_{a,n-1}} \left(\sum_{i_\alpha=1}^{d_\alpha} (\varphi_R^\mu)_{i_\alpha i_\alpha}^{\alpha\alpha} (a, n-1) \right). \quad (19)$$

Finally using Corollary 14 from Appendix B we obtain the second statement of the proposition. \square

3 Deterministic version of the protocol

Theorem 3. *The fidelity for Port Based Teleportation is given by the following formula*

$$F = \frac{1}{d^{N+2}} \sum_{\alpha \vdash n-2} \left(\sum_{\mu \in \alpha} \sqrt{d_\mu m_\mu} \right)^2, \quad (20)$$

where sums over α and μ are taken, whenever number of rows in corresponding Young diagrams is not greater than the dimension of the local Hilbert space d .

¹In overall paper by m_α, m_μ we denote multiplicities in the natural representation.

Proof. From [10] we know, that the fidelity in PBT is given by the following formula

$$F = \frac{1}{d^2} \sum_{a=1}^N \text{Tr} \left[\varrho_a \rho^{-1/2} \varrho_a \rho^{-1/2} \right] = \frac{N}{d^{N+2}} \text{Tr} \left[V^{t_n}(n-1, n) \eta^{-1/2} V^{t_n}(n-1, n) \eta^{-1/2} \right]. \quad (21)$$

Since every term in (21) does not depend on the index a , so is enough to compute quantity (21) for the simplest case, when $a = N$, then $V^{t_n}(n-1, n) = \mathbf{1} \otimes P_{n-1, n}^+ \equiv \mathbf{1} \otimes P_+$. Thanks to this, Theorem 1 and Fact 9 from Appendix B we have:

$$\begin{aligned} F &= \frac{N}{d^{N+2}} \text{Tr} \left[(\mathbf{1} \otimes P_+) \eta^{-1/2} (\mathbf{1} \otimes P_+) \eta^{-1/2} \right] \\ &= \frac{N}{d^{N+2}} \sum_{\alpha, \alpha'} \sum_{\substack{\mu \in \alpha \\ \mu' \in \alpha'}} \frac{1}{\sqrt{\gamma_\mu(\alpha)}} \frac{1}{\sqrt{\gamma_{\mu'}(\alpha')}} \text{Tr} \left[V^{t_n}(n-1, n) F_\mu(\alpha) V^{t_n}(n-1, n) F_{\mu'}(\alpha') \right] \\ &= \frac{N}{d^{N+2}} \sum_{\alpha, \alpha'} \sum_{\substack{\mu \in \alpha \\ \mu' \in \alpha'}} \frac{1}{\sqrt{\gamma_\mu(\alpha)}} \frac{1}{\sqrt{\gamma_{\mu'}(\alpha')}} \text{Tr} \left[V^{t_n}(n-1, n) P_\mu M_\alpha V^{t_n}(n-1, n) P_{\mu'} P_{\alpha'} \right] \\ &= \frac{N}{d^{N+2}} \sum_{\alpha, \alpha'} \sum_{\substack{\mu \in \alpha \\ \mu' \in \alpha'}} \frac{1}{\sqrt{\gamma_\mu(\alpha)}} \frac{1}{\sqrt{\gamma_{\mu'}(\alpha')}} \text{Tr} \left[P_{\alpha'} V^{t_n}(n-1, n) P_\mu P_{\alpha'} V^{t_n}(n-1, n) P_{\mu'} \right]. \end{aligned} \quad (22)$$

Now applying Fact 15 to the projector P_μ and making use of Fact 17 we get:

$$\begin{aligned} F &= \frac{N}{d^{N+2}} \sum_{\alpha, \alpha'} \sum_{\substack{\mu \in \alpha \\ \mu' \in \alpha'}} \frac{1}{\sqrt{\gamma_\mu(\alpha)}} \frac{1}{\sqrt{\gamma_{\mu'}(\alpha')}} \times \\ &\times \sum_{a=1}^{n-1} \sum_{ij=1}^{d_\mu} \frac{d_\mu}{(n-1)!} \varphi_{ij}^\mu(a, n-1) \text{Tr} \left[\left(P_\alpha F_{ij}^\mu \otimes \mathbf{1} \right) (\mathbf{1} \otimes P_+) V(a, n-1) (\mathbf{1} \otimes P_+) (P_{\alpha'} \otimes \mathbf{1}) P_{\mu'} \right] \\ &= \frac{N}{d^{N+2}} \sum_{\alpha, \alpha'} \sum_{\substack{\mu \in \alpha \\ \mu' \in \alpha'}} \frac{1}{\sqrt{\gamma_\mu(\alpha)}} \frac{1}{\sqrt{\gamma_{\mu'}(\alpha')}} \sum_{a=1}^{n-1} \sum_{ij=1}^{d_\mu} \frac{d_\mu d^{\delta_{a, n-1}}}{(n-1)!} \varphi_{ij}^\mu(a, n-1) \text{Tr} \left[\left(P_\alpha F_{ij}^\mu P_{\alpha'} \otimes P_+ \right) P_{\mu'} \right] \\ &= \frac{N}{d^{N+2}} \sum_{\alpha, \alpha'} \sum_{\substack{\mu \in \alpha \\ \mu' \in \alpha'}} \frac{1}{\sqrt{\gamma_\mu(\alpha)}} \frac{1}{\sqrt{\gamma_{\mu'}(\alpha')}} \sum_{a=1}^{n-1} \sum_{ij=1}^{d_\mu} \frac{d_\mu d^{\delta_{a, n-1}}}{(n-1)!} \varphi_{ij}^\mu(a, n-1) \text{Tr} \left[\left(P_\alpha F_{ij}^\mu P_{\alpha'} \otimes \mathbf{1} \right) P_{\mu'} \right]. \end{aligned} \quad (23)$$

In the last equality we have used the fact, that $\text{Tr}_n P^+ = \mathbf{1}$, where identity acts on $(n-1)^{\text{th}}$ subsystem. Applying again Lemma 15, this time to operator $P_{\mu'}$ and calculating partial trace over $(n-1)^{\text{th}}$ subsystem:

$$\begin{aligned} F &= \frac{N}{d^{N+2}} \sum_{\alpha, \alpha'} \sum_{\substack{\mu \in \alpha \\ \mu' \in \alpha'}} \frac{1}{\sqrt{\gamma_\mu(\alpha)}} \frac{1}{\sqrt{\gamma_{\mu'}(\alpha')}} \times \\ &\times \sum_{a, b=1}^{n-1} \sum_{ij=1}^{d_\mu} \frac{d_\mu d_{\mu'} d^{\delta_{a, n-1}} d^{\delta_{b, n-1}}}{[(n-1)!]^2} \varphi_{ij}^\mu(a, n-1) \varphi_{kl}^{\mu'}(b, n-1) \text{Tr} \left[\left(P_\alpha F_{ij}^\mu \right) \left(P_{\alpha'} F_{kl}^{\mu'} \right) \right]. \end{aligned} \quad (24)$$

Because we know from expression (77), that operators $F_{ij}^\mu, F_{kl}^{\mu'}$ can be expressed as direct sum of operators E_{st}^β , where $\beta \vdash n-2$, so

$$\text{Tr} \left[\left(P_\alpha F_{ij}^\mu \right) \left(P_{\alpha'} F_{kl}^{\mu'} \right) \right] = \frac{[(n-2)!]^2}{d_\alpha^2} m_\alpha \delta_{\alpha\alpha'} \delta_{li} \delta_{jk}. \quad (25)$$

Substituting (25) into (24) and making simplifications we obtain the following equation:

$$F = \frac{1}{Nd^{N+2}} \sum_\alpha \frac{m_\alpha}{d_\alpha^2} \sum_{\substack{\mu \in \alpha \\ \mu' \in \alpha'}} \frac{d_\mu d_{\mu'}}{\sqrt{\gamma_\mu(\alpha) \gamma_{\mu'}(\alpha)}} \sum_{a,b=1}^{n-1} d^{\delta_{a,n-1}} d^{\delta_{b,n-1}} \sum_{i_\alpha, j_\alpha=1}^{d_\alpha} (\phi_R^\mu)_{i_\alpha j_\alpha}^{\alpha\alpha} (a, n-1) (\phi_R^{\mu'})_{j_\alpha i_\alpha}^{\alpha\alpha} (b, n-1), \quad (26)$$

where we represent all irreps in the PRIR's form defined in Section B. In the next step we transform the last to sums, which may be written as follows

$$\begin{aligned} & \sum_{a,b=1}^{n-1} d^{\delta_{a,n-1}} d^{\delta_{b,n-1}} \sum_{i_\alpha, j_\alpha=1}^{d_\alpha} (\phi_R^\mu)_{i_\alpha j_\alpha}^{\alpha\alpha} (a, n-1) (\phi_R^{\mu'})_{j_\alpha i_\alpha}^{\alpha\alpha} (b, n-1) \\ &= \sum_{i_\alpha, j_\alpha=1}^{d_\alpha} \left\{ \left[\sum_{a=1}^{n-1} d^{\delta_{a,n-1}} (\phi_R^\mu)_{i_\alpha j_\alpha}^{\alpha\alpha} (a, n-1) \right] \left[\sum_{b=1}^{n-1} d^{\delta_{b,n-1}} (\phi_R^{\mu'})_{j_\alpha i_\alpha}^{\alpha\alpha} (b, n-1) \right] \right\} \\ &= \sum_{i_\alpha, j_\alpha=1}^{d_\alpha} \left\{ \left[\sum_{a=1}^{n-2} (\phi_R^\mu)_{i_\alpha j_\alpha}^{\alpha\alpha} (a, n-1) + d \delta_{i_\alpha j_\alpha} \right] \left[\sum_{b=1}^{n-2} (\phi_R^{\mu'})_{j_\alpha i_\alpha}^{\alpha\alpha} (b, n-1) + d \delta_{j_\alpha i_\alpha} \right] \right\}. \end{aligned} \quad (27)$$

Now we use the second statement of Proposition 11, when $m = n-1$, for the sums over a and b , which yields

$$\begin{aligned} & \sum_{i_\alpha, j_\alpha=1}^{d_\alpha} \left\{ \left[\left(\frac{(n-1)(n-2)}{2} \frac{\chi^\mu(12)}{d_\mu} - \frac{(n-2)(n-3)}{2} \frac{\chi^\alpha(12)}{d_\alpha} \right) \delta_{i_\alpha j_\alpha} + d \delta_{i_\alpha j_\alpha} \right] \times \right. \\ & \times \left. \left[\left(\frac{(n-1)(n-2)}{2} \frac{\chi^{\mu'}(12)}{d_{\mu'}} - \frac{(n-2)(n-3)}{2} \frac{\chi^\alpha(12)}{d_\alpha} \right) \delta_{j_\alpha i_\alpha} + d \delta_{j_\alpha i_\alpha} \right] \right\}, \end{aligned} \quad (28)$$

and after simple reordering we get

$$\begin{aligned} & \left[\left(\frac{(n-1)(n-2)}{2} \frac{\chi^\mu(12)}{d_\mu} - \frac{(n-2)(n-3)}{2} \frac{\chi^\alpha(12)}{d_\alpha} \right) + d \right] \times \\ & \left[\left(\frac{(n-1)(n-2)}{2} \frac{\chi^{\mu'}(12)}{d_{\mu'}} - \frac{(n-2)(n-3)}{2} \frac{\chi^\alpha(12)}{d_\alpha} \right) + d \right] \sum_{i_\alpha, j_\alpha=1}^{d_\alpha} \delta_{i_\alpha j_\alpha} \delta_{j_\alpha i_\alpha}. \end{aligned} \quad (29)$$

In equation (29) we recognize inside the square brackets the expression for eigenvalues $\gamma_\mu(\alpha)$ and $\gamma_{\mu'}(\alpha)$, this yields

$$\sum_{a,b=1}^{n-1} d^{\delta_{a,n-1}} d^{\delta_{b,n-1}} \sum_{i_\alpha, j_\alpha=1}^{d_\alpha} (\phi_R^\mu)_{i_\alpha j_\alpha}^{\alpha\alpha} (a, n-1) (\phi_R^{\mu'})_{j_\alpha i_\alpha}^{\alpha\alpha} (b, n-1) = \gamma_\mu(\alpha) \gamma_{\mu'}(\alpha) \dim \varphi^\alpha. \quad (30)$$

Substituting equation (30) into equation (26) we reduce expression for the fidelity F to:

$$F = \frac{1}{Nd^{N+2}} \sum_{\alpha} \frac{m_{\alpha}}{d_{\alpha}} \sum_{\substack{\mu \in \alpha \\ \mu' \in \alpha'}} d_{\mu} d_{\mu'} \sqrt{\gamma_{\mu}(\alpha) \gamma_{\mu'}(\alpha)}. \quad (31)$$

In the last step substituting expression for the eigenvalues $\gamma_{\mu}(\alpha), \gamma_{\mu'}(\alpha)$ given through equation (13) in Theorem 2 we obtain statement of this theorem. \square

4 Probabilistic version of the protocol

To prove optimality of the set of POVMs, we formulate the question as a semidefinite program. From [1] it follows that the optimal POVMs for the probabilistic PBT coincides with the ones for distinguishing the set of states $\{(1/N; \varrho_a)\}_{a=1}^N$. We thus look for a set of POVMs $\{\Pi_a = P_{a,n}^+ \otimes \Theta_{\bar{a}}\}_{a=1}^N$ which would maximize the average success probability

$$p(\Theta_{\bar{a}}) = \frac{1}{d^{N+1}} \sum_{a=1}^N \text{Tr} \Pi_a = \frac{1}{d^{N+1}} \sum_{a=1}^N \text{Tr} \Theta_{\bar{a}} \quad (32)$$

subject to

$$\Theta_{\bar{a}} \geq 0, \quad \sum_{a=1}^N P_{a,n}^+ \otimes \Theta_{\bar{a}} \leq \mathbf{1}_{AB} \quad a = 1, 2, \dots, N. \quad (33)$$

Since our resource state is maximally entangled, the RHS of the second constraint in (33) reduces to identity on AB (see [10]). The main result of this section can be formulated in the following

Theorem 4. *The maximal average success probability in the probabilistic PBT is given by the expression*

$$p = \frac{1}{d^N} \sum_{\alpha} m_{\alpha}^2 \frac{d_{\mu^*}}{m_{\mu^*}}, \quad (34)$$

where μ^* denotes Young diagram obtained from $\alpha \vdash n-2$ by adding single box in a proper way and corresponding to the maximal eigenvalue $\gamma_{\mu^*}(\alpha)$ given through Theorem 2.

We prove above theorem by solving primal and dual semidefinite problem. Define the feasible maximum value of the primal problem:

$$p^* = \max_{\{\Theta_{\bar{a}}\}} p(\Theta_{\bar{a}}). \quad (35)$$

Lemma 5. *The primal is feasible with the optimal value $p^* = \frac{1}{d^N} \sum_{\alpha} m_{\alpha}^2 \frac{d_{\mu^*}}{m_{\mu^*}}$.*

Proof. The symmetries in our problem suggest that we take $\Theta_{\bar{a}}$ as elements of the algebra $\mathbb{C}[S(n-2)]$. Thanks to this $\Theta_{\bar{a}} = \sum_{\alpha} x_{\alpha} P_{\alpha}$, where P_{α} are Young projectors and $x_{\alpha} \in \mathbb{R}_+$ which ensures, that first constraint from (33) is automatically satisfied. Using this argumentation we can rewrite the second constraint from (33) restricted to an irrep labelled by $\alpha \vdash n-2$ as

$$\sum_{a=1}^{n-1} P_{a,n}^+ \otimes \Theta_{\bar{a}}(\alpha) = x_{\alpha} \sum_{a=1}^{n-1} V(a, n-1) P_{n-1,n}^+ \otimes P_{\alpha} V(a, n-1) = \frac{x_{\alpha}}{d} \eta(\alpha), \quad (36)$$

where $\eta(\alpha)$ are defined through Theorem 1. Eigenvalues of the operator $\frac{1}{d}\eta(\alpha)$ are equal to $\frac{1}{d}\gamma_\mu(\alpha)$, where numbers $\gamma_\mu(\alpha)$ are eigenvalues of $\eta(\alpha)$ given in Theorem 2. To ensure that $\forall \alpha \frac{x_\alpha}{d}\eta(\alpha) \leq \mathbf{1}_\alpha$ we take

$$x_\alpha = \min_{\mu \in \alpha} \frac{1}{\frac{1}{d}\gamma_\mu(\alpha)} = d \min_{\mu \in \alpha} \frac{1}{\gamma_\mu(\alpha)}. \quad (37)$$

To obtain the minimum it is enough to insert $\gamma_{\mu^*}(\alpha)$, which is maximal possible eigenvalue of the operator $\eta(\alpha)$ for some particular Young frame $\mu \vdash n-1$ obtained from $\alpha \vdash n-2$ by adding one box in the proper way. Inserting optimal form of operators $\Theta_{\bar{a}}$ into equation (35) we have

$$p^* = \frac{1}{d^{N+1}} \sum_{a=1}^N \text{Tr} \left(\sum_{\alpha} x_\alpha P_\alpha \right) = \frac{N}{d^N} \sum_{\alpha} \frac{1}{\gamma_{\mu^*}(\alpha)} \text{Tr} P_\alpha = \frac{N}{d^N} \sum_{\alpha} \frac{m_\alpha d_\alpha}{\gamma_{\mu^*}(\alpha)}. \quad (38)$$

Using equation (13) for the $\gamma_{\mu^*}(\alpha)$ we obtain

$$p^* = \frac{1}{d^N} \sum_{\alpha} m_\alpha^2 \frac{d_{\mu^*}}{m_{\mu^*}}. \quad (39)$$

□

The dual problem is of minimizing

$$p_\star = \frac{1}{d^{N+1}} \text{Tr} \Omega \quad \text{subject to} \quad \Omega \geq 0, \quad \text{Tr}_{a,n} [P_{a,n}^+ \Omega] \geq \mathbf{1}, \quad (40)$$

where $a = 1, \dots, n-1$, operator Ω acts on n systems the identity $\mathbf{1}$ is defined on $n-2$ systems, and $P_{a,n}^+$ is projector onto maximally entangled state between respective subsystems. In our considerations we assume, that general form of the operator Ω is given as a linear combination of the projectors $F_\mu(\alpha)$ defined in Theorem 1

$$\Omega = \sum_{\alpha} x_{\mu^*}(\alpha) F_{\mu^*}(\alpha), \quad x_{\mu^*}(\alpha) \in \mathbb{C}, \quad (41)$$

where $\mu^* \vdash n-1$ denotes the Young diagram obtained from the Young diagram $\alpha \vdash n-2$ by adding one box in a proper way and corresponds to the maximal eigenvalue $\gamma_{\mu^*}(\alpha)$. From the definition of the constraints and the symmetries of the projectors $F_\mu(\alpha)$ we need the following

Fact 6. *Let $F_\mu(\alpha)$ be the operators given through Theorem 1, and let $V^{t_n}(n-1, n)$ be a permutation operator acting between $(n-1)$ -th and n -th subsystems partially transposed with respect to n -th subsystem, then*

$$\text{Tr}_{n-1,n} [V^{t_n}(n-1, n) F_\mu(\alpha)] = \frac{m_\mu}{m_\alpha} P_\alpha, \quad (42)$$

where numbers m_α, m_μ are multiplicities of the respective irreps and P_α is the Young projector onto subspace labelled by the partition $\alpha \vdash n-2$.

Proof. The operators $F_\mu(\alpha)$ are invariant under action of $S(n-1)$. Hence also under action of $S(n-2)$. The operator $V^{t_n}(n-1, n)$ is invariant under action of $S(n-2)$. Thanks to this the composition $V^{t_n}(n-1, n) F_\mu(\alpha)$ is invariant under action of $S(n-2)$. Therefore by Fact 18 from Appendix C we have $\text{Tr}_{n-1,n} [V^{t_n}(n-1, n) F_\mu(\alpha)] \in \mathbb{C}[S(n-2)]$, and since it is invariant under

action of $S(n-2)$, must be of the form $\bigoplus_{\beta \vdash n-2} y(\beta) P_\beta$, where $y_\beta \in \mathbb{C}$. However by Theorem 1 and Fact 9 from Appendix A we have

$$\begin{aligned} P_\beta V^{t_n}(n-1, n) F_\mu(\alpha) &= P_\beta V^{t_n}(n-1, n) M_\alpha P_\mu = P_\beta V^{t_n}(n-1, n) P_\alpha P_\mu \\ &= \delta_{\alpha\beta} P_\alpha V^{t_n}(n-1, n) F_\mu(\alpha). \end{aligned} \quad (43)$$

This implies, that $\text{Tr}_{n-1, n} [V^{t_n}(n-1, n) F_\mu(\alpha)] = y_\mu(\alpha) P_\alpha$, thus

$$y_\mu(\alpha) = \frac{\text{Tr} [V^{t_n}(n-1, n) F_\mu(\alpha)]}{d_\alpha m_\alpha} = \frac{\text{Tr} [V^{t_n}(n-1, n) M_\alpha P_\mu]}{d_\alpha m_\alpha}. \quad (44)$$

Applying again Fact 9 we have

$$y_\mu(\alpha) = \frac{\text{Tr} [V^{t_n}(n-1, n) P_\alpha P_\mu]}{d_\alpha m_\alpha} = \frac{\text{Tr} [P_\mu (P_\alpha \otimes \mathbf{1})]}{d_\alpha m_\alpha} = \frac{m_\mu}{m_\alpha}. \quad (45)$$

This finishes the proof. \square

Lemma 7. *The dual is feasible with the optimal value $p_\star = \frac{1}{d^N} \sum_\alpha m_\alpha^2 \frac{d_{\mu^*}}{m_{\mu^*}}$.*

Proof. Let us assume that the coefficients $x_{\mu^*}(\alpha)$ given in definition of the operator Ω in equation (41) are of the form $x_{\mu^*}(\alpha) = d \frac{m_\alpha}{m_{\mu^*}}$. Then obviously $\Omega \geq 0$, and by Fact 6 we have the following

$$\begin{aligned} \text{Tr}_{n-1, n} [P_{n-1, n}^+ \Omega] &= \frac{1}{d} \text{Tr} [V^{t_n}(n-1, n) \Omega] = \sum_\alpha \frac{m_\alpha}{m_{\mu^*}} \text{Tr} [V^{t_n}(n-1, n) F_{\mu^*}(\alpha)] \\ &= \sum_\alpha P_\alpha = \mathbf{1}_{1, \dots, n-2}, \end{aligned} \quad (46)$$

where $\mathbf{1}_{1, \dots, n-2}$ denotes identity operator defined on first $n-2$ subsystems. Thanks to this we see, that also second constraint from expression (41) is fulfilled. Finally we can calculate quantity p_\star given through equation (40):

$$p_\star = \frac{1}{d^{N+1}} \text{Tr} \Omega = \frac{1}{d^N} \sum_\alpha \frac{m_\alpha}{m_{\mu^*}} \text{Tr} F_{\mu^*}(\alpha) = \frac{1}{d^N} \sum_\alpha m_\alpha^2 \frac{d_{\mu^*}}{m_{\mu^*}}, \quad (47)$$

since $\text{Tr} F_{\mu^*}(\alpha) = d_{\mu^*} m_\alpha$ by Theorem 1. \square

Thanks to the Lemma 5 and Lemma 7 we are in the position to formulate the following proposition:

Proposition 8. *Because $p^* = p_\star$ we conclude that $\Theta_{\bar{a}} = d \sum_\alpha \frac{1}{\gamma_{\mu^*}(\alpha)} P_\alpha$ for $a = 1, 2, \dots, N$ are the optimal POVMs for the maximally entangled state, so we have proven Theorem 4.*

5 Discussion and Open questions

We found explicit expressions for the performance of all of the variations of the PBT in arbitrary dimension with any number of ports. We expect the tools and techniques introduced here to find a number of applications ranging from the study of quantum states with restricted symmetries and calculating properties of the antiferromagnetic systems to problems in the quantum measurement theory. We now mention some open questions.

Firstly, how to characterize entanglement content in the resource state after one run of any PBT protocol for $d > 2$. We know that in the qubit case the residual entanglement may be recycled to teleport more states [17]. In addition, when probabilistic PBT fails, Alice can nevertheless make use of the standard teleportation protocol reliably [9]. One might wonder if it is possible to similarly utilize the residual entanglement of the resource state in the qudit case. In particular, to show that such higher-dimensional teleportation scheme works would require to extend our analysis to the properties of the subspace S depicted in Fig 2.

For the probabilistic PBT protocol we have shown that the measurement operators are optimal for a fixed resource state of the form $|\Phi^+\rangle^{\otimes N}$. We do not know whether this holds for the deterministic PBT in our case. From [8, 10, 11] we know that for both the KLM scheme and for the PBT protocols teleporting qubits the resource state and the corresponding measurement differ when optimized simultaneously.

Another important problem is to determine the asymptotic performance of the PBT in arbitrary dimension. This presents a challenging task in particular because the asymptotic representation theory for the regime $d/N \rightarrow 0$ is still in its infancy.

Acknowledgments

M.S. is supported by the grant “Mobilność Plus IV”, 1271/MOB/IV/2015/0 from the Polish Ministry of Science and Higher Education. MH and MM are supported by National Science Centre, Poland, grant OPUS 9. 2015/17/B/ST2/01945.

A Facts about operators $F_\mu(\alpha)$

Fact 9. *Let M_α be projector including multiplicities onto α -th irrep of the algebra $\mathcal{A}_n^{t_n}(d)$, let P_α, P_β where $\alpha, \beta \vdash n-2$ be a Young projectors, and let $V^{t_n}(n-1, n)$ be a permutation operator acting between $(n-1)$ -th and n -th subsystems partially transposed with respect to n -th subsystem, then*

$$M_\alpha V^{t_n}(n-1, n) = P_\alpha V^{t_n}(n-1, n), \quad (48)$$

Proof. The proof is based on the results presented in [18]. Namely we know, that operators $V^{t_n}(\sigma)$, where $\sigma \in S(n)$ can be decomposed in every irrep labelled by α in operator basis $\{v_{ij}^{ab}(\alpha)\}$, where $1 \leq a, b \leq n-1$ and $1 \leq i, j \leq \dim \varphi^\alpha$. In particular, when $\sigma = (n-1, n)$ we have:

$$V^{t_n}(n-1, n) = \sum_{\alpha \vdash n-2} \sum_{i,j=1}^{d_\alpha} \varphi_{ij}^\alpha(n-1, n) v_{ij}^{n-1, n-1}(\alpha), \quad (49)$$

but $\sum_{i,j=1}^{d_\alpha} \varphi_{ij}^\alpha(n-1, n) v_{ij}^{n-1, n-1}(\alpha) = M_\alpha V^{t_n}(n-1, n)$ is a restriction of $V^{t_n}(\sigma)$ to irrep labelled by α , so rewriting equation (49)

$$\begin{aligned} M_\alpha V^{t_n}(n-1, n) &= \sum_{i,j=1}^{d_\alpha} \varphi_{ij}^\alpha(n-1, n) v_{ij}^{n-1, n-1}(\alpha) \\ &= \sum_{i,j=1}^{d_\alpha} \varphi_{ij}^\alpha(e) v_{ij}^{n-1, n-1}(\alpha) = \sum_{i=1}^{d_\alpha} v_{ii}^{n-1, n-1}(\alpha). \end{aligned} \quad (50)$$

Using equation (22) from [18] we write:

$$\sum_{i=1}^{d_\alpha} v_{ii}^{n-1, n-1}(\alpha) = \sum_{i=1}^{d_\alpha} \frac{d_\alpha}{(n-2)!} \sum_{\pi \in S(n-2)} \varphi_{ii}^\alpha(\pi^{-1}) V(\pi) V^{t_n}(n-1, n) = P_\alpha V^{t_n}(n-1, n), \quad (51)$$

since $P_\alpha = \sum_{i=1}^{d_\alpha} \frac{d_\alpha}{(n-2)!} \sum_{\pi \in S(n-2)} \varphi_{ii}^\alpha(\pi^{-1}) V(\pi)$. This finishes the proof. \square

B Partially reduced irreducible representations

In this note we derive some properties of the Partially Reduced Irreducible Representations PRIRs. This concept plays a crucial role in the simplification of the representation of the algebra $\mathcal{A}_n^{t_n}(d)$.

Let us consider an arbitrary unitary irrep ϕ^μ of $S(m)$. It can be always unitarily transformed to PRIR ϕ_R^μ such that

$$\forall \pi \in S(m-1) \quad \phi_R^\mu(\pi) = \bigoplus_{\alpha \in \mu} \varphi^\alpha(\pi), \quad (52)$$

where φ^α are irreps of $S(m-1)$. By $\alpha \in \mu$ we understand such Young diagrams α which can be obtained from μ by removing one box in the proper way. We see, that the restriction of the irrep ϕ^μ of $S(m)$ to the subgroup $S(m-1)$ has a block-diagonal form of completely reduced representation, which in matrix notation takes the form

$$\forall \pi \in S(m-1) \quad \phi_R^\mu(\pi) = \left(\delta^{\alpha\beta} \varphi_{i_\alpha j_\alpha}^\alpha(\pi) \right). \quad (53)$$

The block structure of this reduced representation allows us to introduce such a block indexation for PRIR ϕ_R^μ of $S(m)$, which gives

$$\forall \sigma \in S(m) \quad \phi_R^\mu(\sigma) = \left(\phi_{i_\alpha j_\beta}^{\alpha\beta}(\sigma) \right), \quad (54)$$

where the matrices on the diagonal $(\phi_R^\mu)^{\alpha\alpha}(\sigma) = \left(\phi_{i_\alpha j_\alpha}^{\alpha\alpha}(\sigma) \right)$ are of dimension of corresponding irrep φ^α of $S(m-1)$. The off diagonal blocks need not to be square. The matrices $(\phi_R^\mu)^{\alpha\alpha}(\sigma) = \left(\phi_{i_\alpha j_\alpha}^{\alpha\alpha}(\sigma) \right)$ on the diagonal of the matrix $\phi_R^\mu(\sigma)$ have the following important properties:

Proposition 10. *Let $(\phi_R^\mu)^{\alpha\alpha}(\sigma) = \left(\phi_{i_\alpha j_\alpha}^{\alpha\alpha}(\sigma) \right)$ be the matrices on the diagonal of the PRIR matrix $\phi_R^\mu(\sigma)$, then*

$$\forall \alpha \in \mu \quad \varphi^\alpha(\pi) \left(\phi_R^\mu \right)^{\alpha\alpha}(am) \varphi^\alpha(\pi^{-1}) = \left(\phi_R^\mu \right)^{\alpha\alpha}(\pi(a)m), \quad (55)$$

and from this it follows

$$\forall \alpha \in \mu \quad \forall \pi \in S(m-1) \quad \forall a = 1, \dots, m-1 \quad \text{Tr} [(\phi_R^\mu)^{\alpha\alpha}(am)] = \text{Tr} [(\phi_R^\mu)^{\alpha\alpha}(\rho(a)m)], \quad (56)$$

so the trace is constant on the transpositions which naturally indexed the coset $S(m)/S(m-1)$.

Proof. From the composition rule in $S(m)$ we have

$$\forall \pi \in S(m-1) \quad \forall a = 1, \dots, m-1 \quad \pi \circ (am) \circ \pi^{-1} = (\pi(a)m), \quad (57)$$

which implies

$$\forall \rho \in S(m-1) \quad \forall a = 1, \dots, m-1 \quad \phi_R^\mu(\pi) \phi_R^\mu(am) \phi_R^\mu(\pi^{-1}) = \phi_R^\mu(\pi(a)m), \quad (58)$$

where the matrices $\phi_R^\mu(\rho)$, $\phi_R^\mu(\rho^{-1})$ are block-diagonal (see expression (53)). From multiplication rule of block diagonal matrices we get that the equation (58) in the irrep ϕ_R^μ yields the following equations for its diagonal blocks determined by the irrep's φ^α of $S(m-1)$

$$\forall \alpha \in \mu \quad \varphi^\alpha(\pi) \left((\phi_R^\mu)^{\alpha\alpha}(am) \right) \varphi^\alpha(\pi^{-1}) = \left((\phi_R^\mu)^{\alpha\alpha}(\pi(a)m) \right). \quad (59)$$

Taking the trace, in $\mathbb{M}(\dim \varphi^\alpha, \mathbb{C})$, on this equation we get the second statement of the Proposition. \square

Further we have the following sum rules

Proposition 11. *The PRIR ϕ_R^μ of $S(m)$ satisfies the following sum rules*

$$\sum_{a=1}^{m-1} (\phi_R^\mu)(am) = \frac{m(m-1)}{2} \frac{\chi^\mu(12)}{d_\mu} \mathbb{1}_{\phi^\mu} - \bigoplus_{\alpha \in \mu} \frac{(m-1)(m-2)}{2} \frac{\chi^\alpha(12)}{d_\alpha} \mathbb{1}_{\varphi^\alpha}, \quad (60)$$

which implies that for the diagonal blocks we have

$$\forall \alpha \in \mu \quad \sum_{a=1}^{m-1} (\phi_R^\mu)^{\alpha\alpha}(am) = \left[\frac{m(m-1)}{2} \frac{\chi^\mu(12)}{d_\mu} - \frac{(m-1)(m-2)}{2} \frac{\chi^\alpha(12)}{d_\alpha} \right] \mathbb{1}_{\varphi^\alpha}. \quad (61)$$

Proof. The starting point is the classical equation

$$\sum_{(ab) \in S(m)} (\phi_R^\mu)(ab) = \frac{m(m-1)}{2} \frac{\chi^\mu(12)}{d_\mu} \mathbb{1}_{\phi^\mu}, \quad (62)$$

which holds for any irrep of $S(m)$. We rewrite *LHS* of equation (62) separating the terms in $S(m-1)$

$$\begin{aligned} \sum_{(ab) \in S(m)} (\phi_R^\mu)(ab) &= \sum_{a=1}^{m-1} (\phi_R^\mu)(am) + \sum_{(cd) \in S(m-1)} (\phi_R^\mu)(cd) \\ &= \sum_{a=1}^{m-1} (\phi_R^\mu)(am) + \bigoplus_{\alpha \in \mu} \sum_{(cd) \in S(m-1)} \varphi^\alpha(cd). \end{aligned} \quad (63)$$

Now we use one more equation (62) to each irrep φ^α in the direct sum of RHS in equation (63) we get

$$\sum_{(ab) \in S(m)} (\phi_R^\mu)(ab) = \sum_{a=1}^{m-1} (\phi_R^\mu)(am) + \bigoplus_{\alpha \in \mu} \frac{(m-1)(m-2)}{2} \frac{\chi^\alpha(12)}{d_\alpha} \mathbb{1}_{\varphi^\alpha}. \quad (64)$$

This equation together with expression (62) gives the first statement of the Proposition. \square

Remark 12. Equation (60) in Proposition 11 may be written in a more explicit form as follows:

$$\forall \alpha \in \mu \quad \sum_{a=1}^{m-1} (\phi_R^\mu)^{\alpha\alpha}_{i_\alpha j_\alpha}(am) = \left[\frac{m(m-1)}{2} \frac{\chi^\mu(12)}{d_\mu} - \frac{(m-1)(m-2)}{2} \frac{\chi^\alpha(12)}{d_\alpha} \right] \delta_{i_\alpha j_\alpha}, \quad (65)$$

where $i_\alpha, j_\alpha = 1, \dots, \dim \varphi^\alpha$.

We have one more sum rule, which plays a role of the standard orthogonality relation for irreps. Namely we have the following:

Proposition 13. The PRIR ϕ_R^μ of $S(m)$ satisfies the following bilinear sum rule

$$\forall \alpha, \beta, \gamma \in \mu \quad \sum_{a=1}^m \sum_{k_\beta=1}^{\dim \varphi^\beta} (\phi_R^\mu)^{\alpha\beta}_{i_\alpha k_\beta}(am) (\phi_R^\mu)^{\beta\gamma}_{k_\beta j_\gamma}(am) = m \frac{d_\beta}{d_\mu} \delta^{\alpha\gamma} \delta_{i_\alpha j_\gamma}, \quad (66)$$

where α, β, γ are irreps of $S(m-1)$ contained in the irrep μ of $S(m)$.

Proof. The proof is based on the standard orthogonality relations for irreps, which in PRIR notation take the following form

$$\forall \alpha, \beta, \gamma \in \mu \quad \sum_{\sigma \in S(m)} (\phi_R^\mu)^{\alpha\beta}_{i_\alpha k_\beta}(\sigma^{-1}) (\phi_R^\mu)^{\beta\gamma}_{k_\beta j_\gamma}(\sigma) = \frac{m!}{d_\mu} \delta^{\alpha\gamma} \delta_{i_\alpha j_\gamma}, \quad (67)$$

for any irreps α, β, γ of the group $S(m-1)$ which are contained in the irrep μ of $S(m)$. On the other hand we may rewrite the LHS of the above equation as follows

$$LHS = \sum_{a=1}^m \sum_{\pi \in S(m-1)} \sum_{\xi, \theta \in \mu} \sum_{p_\xi, q_\theta} (\phi_R^\mu)^{\alpha\xi}_{i_\alpha p_\xi}(am) (\phi_R^\mu)^{\xi\beta}_{p_\xi k_\beta}(\pi^{-1}) (\phi_R^\mu)^{\beta\theta}_{k_\beta q_\theta}(\pi) (\phi_R^\mu)^{\theta\gamma}_{q_\theta j_\gamma}(am). \quad (68)$$

Taking into account equation (53) we obtain

$$LHS = \sum_{a=1}^m \sum_{\pi \in S(m-1)} \sum_{p_\beta, q_\beta} (\phi_R^\mu)^{\alpha\beta}_{i_\alpha p_\beta}(am) \varphi^\beta_{p_\beta k_\beta}(\pi^{-1}) \varphi^\beta_{k_\beta q_\beta}(\pi) (\phi_R^\mu)^{\beta\gamma}_{q_\beta j_\gamma}(am), \quad (69)$$

next applying the orthogonality relations for irreps φ^β of $S(m-1)$ we get

$$LHS = \frac{(m-1)!}{d_\beta} \sum_{a=1}^m \sum_{p_\beta}^{d_\beta} (\phi_R^\mu)^{\alpha\beta}_{i_\alpha p_\beta}(am) (\phi_R^\mu)^{\beta\gamma}_{p_\beta j_\gamma}(am). \quad (70)$$

Now comparing this with the RHS of the equation (67) we obtain the statement of the proposition. \square

As a corollary from Propositions 10 and Proposition 11 we get

Corollary 14.

$$\forall \alpha \in \mu \quad \forall a = 1, \dots, m-1 \quad \text{Tr} [(\phi_R^\mu)^{\alpha\alpha}(am)] = \frac{m}{2} \frac{d_\alpha}{d_\beta} \chi^\mu(12) - \frac{m-2}{2} \chi^\alpha(12). \quad (71)$$

C Auxiliary facts

Let us define the following set of permutations

$$\Sigma_a = \{\sigma \in S(n-1) : \sigma(a) = n-1\}, \quad \text{then we have} \quad S(n-1) = \bigcup_{a=1}^{n-1} \Sigma_a. \quad (72)$$

Now we see, that for every $\sigma \in \Sigma_a$ permutation $\sigma \circ (a, n-1)$ belongs to $S(n-2)$, since $(\sigma \circ (a, n-1))(n-1) = n-1$. Such property allows us to rewrite Young projectors P_μ , where $\mu \vdash n-1$ in more convenient form, namely we have the following:

Fact 15. *Young projector P_μ , where $\mu \vdash n-1$ can be written as*

$$P_\mu = \frac{d_\mu}{(n-1)!} \sum_{a=1}^{n-1} \sum_{i,j=1}^{d_\mu} \varphi_{ij}^\mu(a, n-1) V(a, n-1) K_{ij}^\mu, \quad (73)$$

where

$$K_{ij}^\mu = \sum_{\pi \in S(n-2)} \varphi_{ji}^\mu(\pi^{-1}) V(\pi). \quad (74)$$

By $\varphi_{ij}^\mu(a, n-1), \varphi_{ji}^\mu(\pi^{-1})$ we denote matrix elements of irreducible representations labelled by partition μ for the permutations $(a, n-1), \pi^{-1}$ respectively. Note that in the equation (74) we compute matrix elements of irreducible representations for partition $\mu \vdash n-1$, but over subgroup $S(n-2) \subset S(n-1)$.

Proof. Proof is based on straightforward calculations and observations summarized in the formula (72). We have the following chain of equalities:

$$\begin{aligned} P_\mu &= \frac{d_\mu}{(n-1)!} \sum_{\sigma \in S(n-1)} \chi^\mu(\sigma^{-1}) V(\sigma) = \frac{d_\mu}{(n-1)!} \sum_{a=1}^{n-1} \sum_{\pi \in S(n-2)} \chi^\mu((a, n-1) \circ \pi^{-1}) V((a, n-1) \circ \pi) \\ &= d_\mu \sum_{a=1}^{n-1} \sum_{i,j=1}^{d_\mu} \varphi_{ij}^\mu(a, n-1) V(a, n-1) \left(\frac{1}{(n-1)!} \sum_{\pi \in S(n-2)} \varphi_{ji}^\mu(\pi^{-1}) V(\pi) \right) \\ &= d_\mu \sum_{a=1}^{n-1} \sum_{i,j=1}^{d_\mu} \varphi_{ij}^\mu(a, n-1) V(a, n-1) K_{ij}^\mu. \end{aligned} \quad (75)$$

□

Every irreducible block labelled by $\mu \vdash n - 1$ can be decomposed as a direct sum of smaller irreducible blocks labelled by partitions $\beta \vdash n - 2$. Every such partition β is obtained by removing from μ single box in the proper way. This together with the notion of PRIRs defined in Appendix A allows us to decompose every K_{ij}^μ from Fact 15 as

$$K_{ij}^\mu = \bigoplus_{\beta=\mu-\square} \sum_{\pi \in S(n-2)} (\varphi_R^\mu)_{i_\beta j_\beta}^{\beta\beta} (\pi^{-1}) V(\pi). \quad (76)$$

Moreover every operator K_{ij}^μ can be expressed in terms of the projectors E_{ij}^β as

$$K_{ij}^\mu = \bigoplus_{\beta=\mu-\square} \frac{(n-2)!}{d_\beta} E_{i_\beta j_\beta}^\beta. \quad (77)$$

Fact 16. Suppose, that we are given with irreducible representation labelled by $\mu \vdash n - 1$, then for every swap operator $V(k, n - 1)$ between k^{th} and $(n - 1)^{\text{th}}$ subsystem, and Young projector P_μ we have

$$\sum_{k=1}^{n-1} V(k, n - 1) P_\mu V(k, n - 1) = (n - 1) P_\mu. \quad (78)$$

Proof. We know, that every Young projector associated with irreducible representation μ can be written as

$$P_\mu = \frac{d_\mu}{(n-1)!} \sum_{\sigma \in S(n-1)} \chi^\mu(\sigma^{-1}) V(\sigma), \quad (79)$$

where $\chi^\mu(\sigma^{-1})$ is the character of irreducible representation μ calculated on the element $\sigma^{-1} \in S(n - 1)$, and $V(\sigma)$ is the permutation operator. Since operator P_μ belongs to the centre of the algebra $\mathbb{C}[S(n - 1)]$ it commutes with all elements $V(\sigma) \in \mathbb{C}[S(n - 1)]$, where $\sigma \in S(n - 1)$ in particular with $V(k, n - 1) \in \mathbb{C}[S(n - 1)]$ for $k = 1, \dots, n - 1$. This finishes the proof. \square

Fact 17. Let us denote by P_+ projector onto unnormalized maximally entangled state $|\psi^+\rangle = \sum_i |ii\rangle$ between $(n - 1)^{\text{th}}$ and n^{th} subsystem, then:

$$(\mathbf{1} \otimes P_+) V(k, n - 1) (\mathbf{1} \otimes P_+) = \begin{cases} d(\mathbf{1} \otimes P_+) & \text{if } k = n - 1, \\ \mathbf{1} \otimes P_+ & \text{if } k = 1, \dots, n - 2. \end{cases} \quad (80)$$

In above by $V(k, n - 1)$ we denote swap operator between k^{th} and $(n - 1)^{\text{th}}$ subsystem respectively, and by d dimension of the local Hilbert space.

Proof. For $k = n - 1$ we have simply $(\mathbf{1} \otimes P_+)^2 = d(\mathbf{1} \otimes P_+)$, since P_+ is unnormalized. Now we have to prove the second case from the formula (80):

$$\begin{aligned} (\mathbf{1} \otimes P_+) V(k, n - 1) (\mathbf{1} \otimes P_+) &= \left(\sum_{j_n, j_{n-1}=1}^d \mathbf{1}_1 \otimes \dots \otimes \mathbf{1}_k \otimes \dots \otimes |j_n\rangle\langle j_{n-1}| \otimes |j_n\rangle\langle j_{n-1}| \right) \\ &\times \left(\sum_{i_k, i_{n-1}=1}^d \mathbf{1}_1 \otimes \dots \otimes \mathbf{1}_{k-1} \otimes |i_{n-1}\rangle\langle i_k| \otimes \dots \otimes |i_k\rangle\langle i_{n-1}| \otimes \mathbf{1}_n \right) \\ &\times \left(\sum_{l_{n-1}, l_n=1}^d \mathbf{1}_1 \otimes \dots \otimes \mathbf{1}_k \otimes \dots \otimes |l_n\rangle\langle l_{n-1}| \otimes |l_n\rangle\langle l_{n-1}| \right) \end{aligned} \quad (81)$$

$$\begin{aligned}
&= \sum_{\substack{j_n, j_{n-1}=1 \\ i_k, i_{n-1}=1 \\ l_{n-1}, l_n=1}}^d \mathbf{1}_1 \otimes \cdots \otimes \mathbf{1}_{k-1} \otimes |i_{n-1}\rangle\langle i_k| \otimes \cdots \otimes |j_n\rangle\langle j_{n-1}| i_k\rangle\langle i_{n-1}| l_n\rangle\langle l_{n-1}| \otimes |j_n\rangle\langle j_{n-1}| l_n\rangle\langle l_{n-1}| \\
&= \sum_{i_k, j_n, l_{n-1}=1}^d \mathbf{1}_1 \otimes \cdots \otimes \mathbf{1}_{k-1} \otimes |i_k\rangle\langle i_k| \otimes \cdots \otimes |j_n\rangle\langle l_{n-1}| \otimes |j_n\rangle\langle l_{n-1}| = \mathbf{1} \otimes P_+.
\end{aligned} \tag{82}$$

□

Fact 18. For an arbitrary element X of algebra $\mathcal{A}_n^{t_n}(d)$, $\text{Tr}_n X \in \mathbb{C}[S(n-1)]$.

Proof. From [14, 18] we know, that algebra $\mathcal{A}_n^{t_n}(d)$ is spanned by the partially transposed permutation operators $V^{t_n}(\sigma)$, where $\sigma \in S(n)$. Let us take an arbitrary operator A defined on $n-1$ subsystems, then we can write

$$\text{Tr} [V^{t_n}(\sigma)A \otimes \mathbf{1}_n] = \text{Tr} [V(\sigma)A \otimes \mathbf{1}_n^{t_n}], \tag{83}$$

where $\mathbf{1}_n$ is the identity operator on last system, and t_n denotes standard transposition operation on last n -th system. Thanks to this we can write $\text{Tr}_n [V^{t_n}(\sigma)] = \text{Tr} [V(\sigma)]$, but $\text{Tr} [V(\sigma)]$ for $\sigma \in S(n)$ belongs to $\mathbb{C}[S(n-1)]$, so we have $\text{Tr}_n [V^{t_n}(\sigma)] \in \mathbb{C}[S(n-1)]$. □

D Operators E_{ij}^α

In this section we briefly remind some properties of the algebra generated by a given complex finite dimensional representation of the finite group G . The content of this section can be found in the standard books on representation theory of finite groups and algebras, for example in [5].

Any complex finite-dimensional representation $D : G \rightarrow \text{Hom}(V)$ of the finite group G , where V is a complex linear space, generates an algebra $A_V[G] \subset \text{Hom}(V)$ which is isomorphic to the group algebra $\mathbb{C}[G]$ if the representation D is faithful. Obviously

$$A_V[G] = \text{span}_{\mathbb{C}}\{D(g), \quad g \in G\}. \tag{84}$$

If the operators $D(g)$ are linearly independent, then they form a basis of the algebra $A_V[G]$ and $\dim A_V[G] = |G|$. It is also possible, using matrix irreducible representations, to construct a new basis which has remarkable properties, very useful in applications of representation theory. Below we describe this construction.

Notation 19. Let G be a finite group of order $|G| = n$ which has r classes of conjugated elements. Then G has exactly r inequivalent, irreducible representations, in particular G has exactly r inequivalent, irreducible matrix representations. Let

$$D^\alpha : G \rightarrow \text{Hom}(V^\alpha), \quad \alpha = 1, 2, \dots, r, \quad \dim V^\alpha = d_\alpha \tag{85}$$

be all inequivalent, irreducible representations of G and let choose these representations to be all unitary (always possible) i.e.

$$D^\alpha(g) = (D_{ij}^\alpha(g)), \quad \text{and} \quad (D_{ij}^\alpha(g))^\dagger = (D_{ij}^\alpha(g))^{-1}, \tag{86}$$

where $i, j = 1, 2, \dots, d_\alpha$.

The matrix elements $D_{ij}^\alpha(g)$ will play a crucial role in the following.

Definition 20. Let $D : G \rightarrow \text{Hom}(V)$ be an unitary representation of a finite group G such that the operators $D(g)$, $g \in G$ are linearly independent i.e. $\dim A_V[G] = |G|$ and let $D^\alpha : G \rightarrow \text{Hom}(V^\alpha)$ be all inequivalent, irreducible representations of G described in Notation 19 above. Define

$$E_{ij}^\alpha = \frac{d_\alpha}{n} \sum_{g \in G} D_{ji}^\alpha(g^{-1}) D(g), \quad (87)$$

where $\alpha = 1, 2, \dots, r$, $i, j = 1, 2, \dots, d_\alpha$, $E_{ij}^\alpha \in A_V[G] \subset \text{Hom}(V)$.

The operators have the following properties:

Theorem 21. 1. There are exactly $|G| = n$ nonzero operators E_{ij}^α and

$$D(g) = \sum_{ij\alpha} D_{ij}^\alpha(g) E_{ij}^\alpha \quad (88)$$

2. the operators E_{ij}^α are orthogonal with respect to the Hilbert-Schmidt scalar product in the space $\text{Hom}(V)$.

$$(E_{ij}^\alpha, E_{kl}^\beta) = \text{Tr}((E_{ij}^\alpha)^\dagger E_{kl}^\beta) = k_\alpha \delta^{\alpha\beta} \delta_{ik} \delta_{jl}, \quad k_\alpha \geq 1, \quad (89)$$

where k_α is equal to the multiplicity of the irreducible representation D^α in D and it does not depend on $i, j = 1, 2, \dots, d_\alpha$

3. the operators E_{ij}^α satisfy the following composition rule

$$E_{ij}^\alpha E_{kl}^\beta = \delta^{\alpha\beta} \delta_{jk} E_{il}^\alpha, \quad (90)$$

in particular E_{ii}^α are orthogonal projections.

Remark 22. From point 2) of above theorem it follows that the equations

$$E_{ij}^\alpha = \frac{d_\alpha}{n} \sum_{g \in G} D_{ji}^\alpha(g^{-1}) D(g) \quad (91)$$

describe transformation of orthogonalization of operators $D(g)$, $g \in G$ in the space $\text{Hom}(V)$ with the Hilbert-Schmidt scalar product.

The operators E_{ii}^α are not only orthogonal projections onto their proper subspaces in V but they are also orthogonal with respect to the Hilbert-Schmidt scalar product in the space $\text{Hom}(V)$. The basis $\{E_{ij}^\alpha\}$ plays essential role when $D : G \rightarrow \mathbb{C}[G]$ is the regular representation. In this case the properties of the basis $\{E_{ij}^\alpha\}$ expresses the well-known fact that the group algebra $\mathbb{C}[G]$ is a direct sum of simple matrix algebras generated by the irreducible representations of the group G . It is always possible to construct the operators E_{ij}^α even if the operators $D(g)$ are not linearly independent but in this case some of them will be zero.

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